# Instanton interactions and Borel summability

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INNOVATIVE ECONOMY
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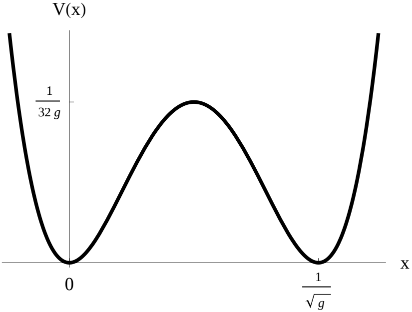
# Plan of the seminar

# Anharmonic double well potential

$$V(x) = \frac{1}{2}x^2(1-\sqrt{g}x)^2$$




Contributions to energy:

- perturbative (Rayleigh Schrödinger perturbation theory)
- nonperturbative (instantons)

# Perturbative expansion

$$H\psi = E\psi \quad \longrightarrow \quad E = \sum_n \epsilon_n g^n$$

[Bender, Wu]: $\epsilon_n \approx -0.95 \times 3^n n!$

Borel transform:

$$\mathcal{B}(t) = \sum_n \frac{1}{n!}\epsilon_n t^n$$

convergent for $|t| < 1/3$

Inverse Borel transform:

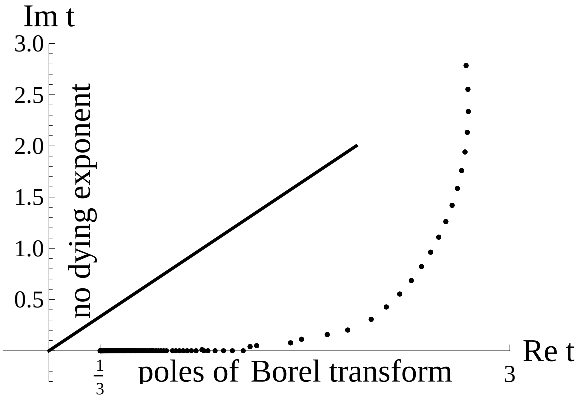
$$E_{Borel}(g) = \frac{1}{g}\int_0^\infty dt e^{-t/g}\mathcal{B}(t)$$

needs $\mathcal{B}(t)$ on whole positive axis $\Rightarrow$ make analytical continuation

# Analytical continuation

Approximate analytical continuation by Padé approximant.
Analytical continuation has a cut at $(1/3, \infty)$.

$\Rightarrow$ Integrate along a contour.




Energy has imaginary part.

Integration on contour with *Im* $t < 0$ gives conjugate energy.

Ambiguity!

# Semiclassical approximation in Euclidean space

In Euclidean space: $$e^{-ET} \propto \int \mathcal{D}[x(\tau)] e^{-S[x(\tau)]}$$

$\Rightarrow$ make saddle point approximation

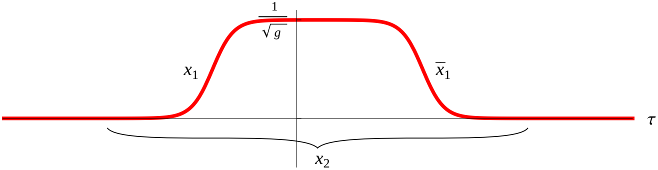
Instantons - classical trajectories in Euclidean space (saddle points).

Action of one instanton:

$$S[x_1(\tau)] = 1/6g$$

Action of instanton - antiinstanton trajectory [Bogomolny]:

$$S[x_2(\tau)] = 2S[x_1(\tau)] - 2e^{-|\tau_1 - \tau_2|}/g$$



$$E_0 = \frac{1}{2} - \frac{1}{\sqrt{g\pi}} e^{-1/6g}$$ - independent instantons

$$+ \frac{1}{g\pi} e^{-1/3g} (\gamma + \ln(-2/g))$$ - corrections from interactions

$$E_1 = \frac{1}{2} + \frac{1}{\sqrt{g\pi}} e^{-1/6g} + \frac{1}{g\pi} e^{-1/3g} (\gamma + \ln(-2/g))$$

$E = \frac{1}{2}(E_0 + E_1)$ has an imaginary ambiguity coming from logarithm!

$$E = \frac{1}{2} + \frac{1}{g\pi} e^{-1/3g} (\gamma + \ln(2/g) \pm i\pi)$$

# Total energy

$$E = \sum_n a_n g^n + \frac{1}{g\pi} e^{-1/3g} (\gamma + \ln(-2/g))$$
$$= E_{Borel} + \delta_2 E$$

Questions

- Do ambiguities of the two contributions cancel?
- Is $Re\delta_2 E$ big enough to be seen in numerics?

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Answers

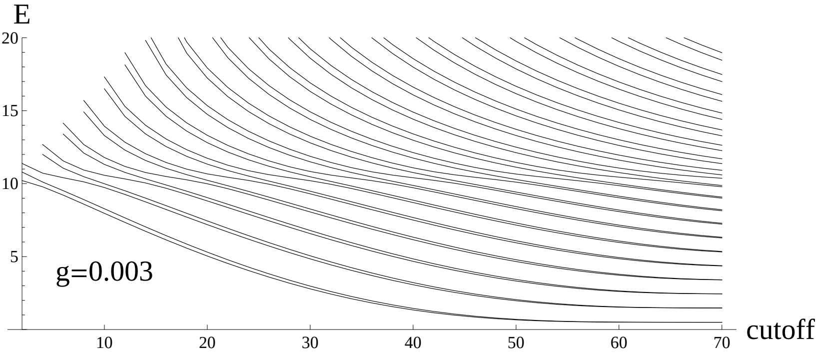
- Yes
- Yes

# Cut Fock space method

- express Hamiltonian as a matrix:

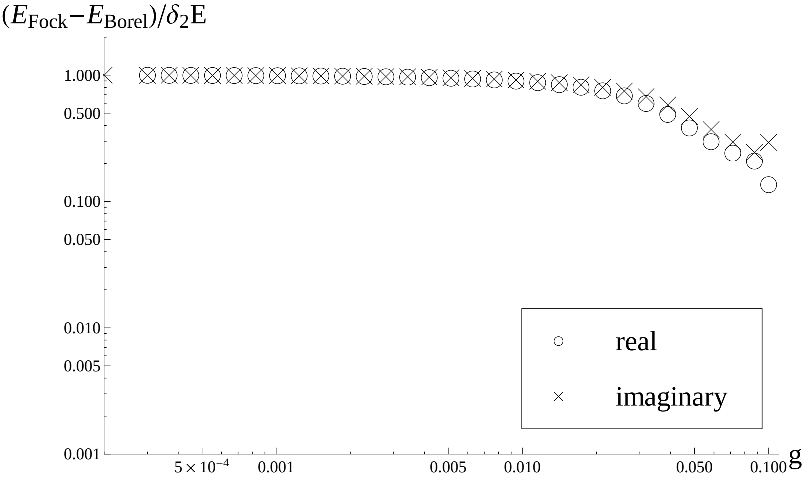
$$(H)_{m,n} = \langle m | \frac{1}{2} P^2 + V(X) | n \rangle$$

- introduce cutoff to get finite matrix
- eigenvalues approximate energies [Wosiek]



# High precision comparison

without $\delta_2 E$



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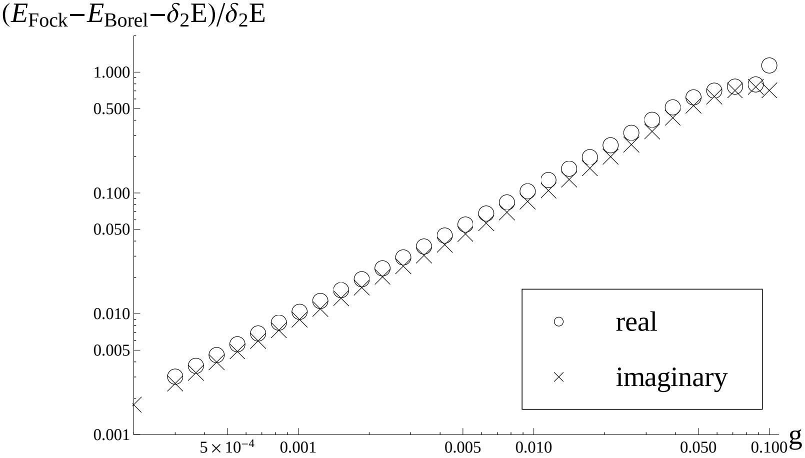


Next correction is seen!

# Summary

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- but only few $c_n$ are known
- next corrections: 3–instanton interactions

# Literature


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